Algebra I : Ist semester 2005
B.Math. Hons.Ist year
Backpaper examination

Q 1.
For a prime p, show that exactly half of the elements in the group $\mathbf{Z}_p^*$ are squares. You may assume that this group of nonzero integers mod p under multiplication mod p, is cyclic.

Q 2.
If H is any subgroup of finite index in a group G, use the action of G on the set of left cosets of H to prove that H contains a normal subgroup of finite index.

Q 3.
Prove that S_n is generated by $(1, 2)$ and $(1, 2, \cdots, n)$.

Q 4.
If P is a p-Sylow subgroup of a finite group G, and if N denotes the normaliser of P in G, prove that the normaliser of N in G is N itself.
Hint : You may use Sylow's second theorem for the group N.

Q 5.
Define the product IJ of two ideals I, J in a commutative ring. Prove that

$$IJ \subset I \cap J \subset I, J \subset I + J.$$

Give examples of ideals I, J in $\mathbf{Z}$ to show that each of these inclusions can be proper.

Q 6.
Let R be a commutative ring containing identity 1 and suppose S is a multiplicatively closed subset. Let P is an ideal of R which does not intersect S and has the property that any ideal of R containing P properly must intersect S. Prove that P is a prime ideal.

Q 7.
In the (noncommutative) ring $M_2(\mathbf{R})$ of 2×2 matrices over $\mathbf{R}$, give examples of left ideals which are not right ideals and examples of right ideals which are not left ideals.